Halos of Unified Dark Matter Scalar Field

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Abstract. We investigate the static and spherically symmetric solutions of Einstein's equations for a scalar field with non-canonical kinetic term, assumed to provide both the dark matter and dark energy components of the Universe. In particular, we give a prescription to obtain solutions (dark halos) whose rotation curve $v_c(r)$ is in good agreement with observational data. We show that there exist suitable scalar field Lagrangians that allow to describe the cosmological background evolution and the static solutions with a single dark fluid.

1. Introduction

The confidence region of cosmological parameters emerging from the analysis of data from type Ia Supernovae (SNIa), Cosmic Microwave Background (CMB) anisotropies and the large scale structure of the Universe, suggests that two dark components govern the dynamics of the present universe. These components are the Dark Matter (DM), responsible for structure formation, and an additional Dark Energy (DE) component that drives the cosmic acceleration observed at present. In this paper we focus on Unified models of Dark Matter and dark energy (UDM) that can provide an alternative to our interpretation of the nature of the dark components of our universe. These models have the advantage over the DM + DE models (e.g. ΛCDM) that one can describe the dynamics of the universe with a single dark fluid which triggers the accelerated expansion at late times and is also the one which has to cluster in order to produce the structures we see today. However, the viability of UDM models strongly depends on the value of the effective speed of sound c_s [1, 2, 3], which has to be small enough to allow structure formation [4, 5, 6] and to reproduce the observed pattern of CMB temperature anisotropies [1, 7, 4, 8].

Several adiabatic or, equivalently, purely kinetic models have been investigated in the literature. For example, the generalized Chaplygin gas ([9, 10, 11] (see also [12]), the Scherrer [13] and generalized Scherrer [7] solutions, the single dark perfect fluid with a simple 2-parameter barotropic equation of state [14], or the homogeneous scalar field deduced from the galactic halo space-times [15].

Moreover, one can build up scalar field models for which the constraint that the Lagrangian is constant along the classical trajectories allows to describe a UDM fluid [7] (see also Ref. [16], for a different approach). Alternative approaches to the unification of DM and DE have been proposed in Ref. [17], in the frame of supersymmetry, and in Ref. [18], in connection with the solution of the strong CP problem.

One could easily reinterpret UDM models based on a scalar field Lagrangian in terms of – generally non-adiabatic – fluids [19, 20].

A complete analysis of UDM models should necessarily include the study of static solutions of Einstein's field equations. This is complementary to the study of cosmological background solutions and would allow to impose further constraints to the Lagrangian of UDM models. The authors of Refs. [21] and [15] have studied spherically symmetric and static configuration for k-essence models. In particular, they studied models where the rotation velocity becomes flat (at least) at large radii of the halo. In these models the scalar field pressure is not small compared to the mass-energy density, similarly to what found in the study of general fluids in Refs. [22, 23, 24, 25], and the Einstein's equations of motion do not reduce to the equations of Newtonian gravity. Further alternative models have been considered, even with a canonical kinetic term in

the Lagrangian, that describe dark matter halos in terms of bosonic scalar fields, see e.g. Refs. [26, 27, 28, 29, 30].
In this paper we assume that our scalar field configurations only depend on the radial direction. Three main results are achieved. First, we are able to find a purely kinetic Lagrangian which allows simultaneously to provide a flat rotation curve and to realize a unified model of dark matter and dark energy on cosmological scales. Second, we have found an invariance property of the expression for the halo rotation curve. This allows to find purely kinetic Lagrangians that reproduce the same rotation curves that are obtained starting from a given density profile within the standard Cold Dark Matter (CDM) paradigm. Finally, we consider a more general class of models with non-purely kinetic Lagrangians. In this case we extend to the static and spherically symmetric space-time metric the procedure used in Ref. [7] to find UDM solutions in a cosmological setting. Such a procedure requires that the Lagrangain is constant along the classical trajectories; we are thus able to provide the conditions to obtain reasonable rotation curves within a UDM model of the type discussed in Ref. [7].

The plan of the paper is as follows. In Section 2 we provide the general framework for the study of static spherically symmetric solutions in UDM models. Section 3 is devoted to the general analysis of purely kinetic Lagrangians, while in Section 4 we analyze more general models with non-canonical kinetic term. Our main conclusions are drawn in Section 5. In the Appendix, for completeness we provide the spherical collapse top-hat solution for UDM models based on purely kinetic scalar-field Lagrangians, which allow to connect the cosmological solutions to the static configurations studied here.

2. Static solutions in Unified Dark Matter scalar field models

We consider the following action

$$S = S_G + S_\varphi + S_b = \int d^4x\sqrt{-g}\left[\frac{R}{2} + \mathcal{L}(\varphi, X)\right] + S_b \qquad (1)$$

where S_b describes the baryonic matter and

$$X = -\frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi\ . \qquad (2)$$

We use units such that $8\pi G = c = 1$ and signature $(-,+,+,+)$; greek indices run over space-time dimensions, while latin indices label spatial coordinates.
The energy-momentum tensor of the scalar field φ is

$$T^\varphi_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\varphi}{\delta g^{\mu\nu}} = \frac{\partial\mathcal{L}(\varphi, X)}{\partial X}\nabla_\mu\varphi\nabla_\nu\varphi + \mathcal{L}(\varphi, X)g_{\mu\nu}\,, \qquad (3)$$

and its equation of motion reads

$$\nabla^\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right] = \frac{\partial\mathcal{L}}{\partial\varphi}\ . \qquad (4)$$

We consider a scalar field which is static and spatially inhomogeneous, i.e. such that $X < 0$. In this situation the energy-momentum tensor is not described by a perfect fluid and its stress energy-momentum tensor reads

$$T^{\varphi}_{\mu\nu} = (p_\parallel + \rho) n_\mu n_\nu - \rho g_{\mu\nu} \tag{5}$$

where

$$\rho = -p_\perp = -\mathcal{L} \ , \tag{6}$$

$n_\mu = \nabla_\mu \varphi / \sqrt{-2X}$ and $p_\parallel = \mathcal{L} - 2X \partial\mathcal{L}/\partial X$. In particular, $p_\parallel$ is the pressure in the direction parallel to n_μ whereas $p_\perp$ is the pressure in the direction orthogonal to n_μ. It is simpler to work with a new definition of X. Indeed, defining $X = -\chi$ we have

$$n_\mu = \nabla_\mu \varphi / (2\chi)^{1/2} \tag{7}$$

$$p_\parallel = 2\chi \frac{\partial \rho}{\partial \chi} - \rho \ . \tag{8}$$

Let us consider for simplicity the general static spherically symmetric space-time metric i.e.

$$ds^2 = -\exp\left(2\alpha(r)\right) dt^2 + \exp\left(2\beta(r)\right) dr^2 + r^2 d\Omega^2 \ , \tag{9}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and α and β are two functions that only depend upon r. As the authors of Refs. [21, 15] have shown, it is easy to see that the non-diagonal term T^{rt} vanishes. Therefore φ could be either strictly static or depend only on time. In this paper we study the solutions where φ depends on the radius only.

In the following we will consider some cases where the baryonic content is not negligible in the halo. In this case we will assume that most of the baryons are concentrated within a radius r_b. If we define M_* as the entire mass of the baryonic component then for $r > r_b$ we can simply assume that M_* is concentrated in the center of the halo.

Considering, therefore, the halo for $r > r_b$, starting from the Einstein's equations and the covariant conservation of the stress-energy (or from the equation of motion of the scalar field, Eq. (4)), we obtain

$$\frac{1}{r^2}\left\{1 - \left[r\exp\left(-2\beta\right)\right]'\right\} = \rho \qquad \Longleftrightarrow \frac{dM}{dr} = 4\pi\rho r^2 \ , \tag{10}$$

$$\frac{1}{r^2}\left\{\exp\left[-2(\alpha+\beta)\right]\left[r\exp\left(2\alpha\right)\right]' - 1\right\} = p_\parallel \qquad \Longleftrightarrow \alpha' = \frac{\frac{M+M_*}{8\pi} + \frac{p_\parallel r^3}{2}}{r^2\left[1 - \frac{M+M_*}{4\pi r}\right]} \ , \tag{11}$$

$$\frac{\exp\left[-(\alpha+2\beta)\right]}{r}\left\{\left[r\exp\alpha\right]'\beta' - \left[r\left(\exp\alpha\right)'\right]'\right\} = \rho \ , \tag{12}$$

$$\frac{dp_\parallel}{dR} = -(p_\parallel + \rho) \tag{13}$$

(which are the 00, rr and $\theta\theta$ components of Einstein's equations and the r component of the continuity equation respectively) where $\exp\left(-2\beta(r)\right) = 1 - (M + M_*)/(4\pi r)$ and $R = \ln[r^2 \exp(\alpha(r))]$. Here a prime indicates differentiation with respect to the radius

r.
A first comment is in order here. If *i)* $\beta' = 0$ and *ii)* $\left[r\,(\exp\alpha)'\right]' > 0$, then we can immediately see that $\rho < 0$. These conditions must therefore be avoided when trying to find a reasonable rotation curve. For example, neglecting the baryonic mass, the special case of $\rho = A/r^2$ and $\exp(\alpha) \sim r^m$, where A and m are constants, fall into this case. We thus recover the *no-go* theorem derived in Ref. [15] under the assumption that the rotation curve $v_c \ll 1$ is constant for all r.

The value of the circular velocity v_c is determined by the assumption that a massive test particle is located at $\theta = \pi/2$. We define as massive test particle the object that sends out a luminous signal to the observer who is considered to be stationary and far away from the halo. In this case the value of $v_c = r\phi'$ can be rewritten as

$$v_c^2 = \frac{p_\| r^2/2 + (M + M_*)/(8\pi r)}{1 - \left[p_\| r^2/2 + 3(M + M_*)/(8\pi r)\right]} \;, \tag{14}$$

but when we consider the weak-field limit condition $(M + M_*)/(8\pi r) \ll 1$ and since the rotation velocities of the halo of a spiral galaxy are typically non-relativistic, $v_c \ll 1$, Eq. (14) simplifies to [21]

$$v_c^2 \approx \frac{M + M_*}{8\pi r} + \frac{p_\| r^2}{2} \;. \tag{15}$$

A second comment follows from the fact that the pressure is not small compared to the mass-energy density. In other words we do not require that general relativity reduces to Newtonian gravity (see also Refs. [22, 23, 24, 25]). Notice also that in the region where $v_c \approx$ const. $\ll 1$ it is easy to see that in general $\exp(\alpha) \approx$ const. since from Eqs. (11) and (15) one obtains $r\alpha' \approx v_c^2$.

Finally, let us point out one of our main results. We can see that the relation (15) is invariant under the following transformation

$$\rho \longrightarrow \widetilde{\rho} = \rho + \sigma(r) \qquad\qquad p_\| \longrightarrow \widetilde{p_\|} = p_\| + q(r) \tag{16}$$

if

$$3q(r) + rq(r)' = -\sigma(r) \;, \tag{17}$$

up to a proper choice of some integration constants. Thanks to this transformation we can consider an ensemble of solutions that have the same rotation curve. We will come back to this point in more detail in the next section. Obviously, these solutions have to satisfy the Einstein's equations (10), (11) and (12), and the covariant conservation of the stress-energy (13). Moreover, we will require the validity of the weak energy conditions, $\rho \geq 0$ and $p_\| + \rho \geq 0$, i.e.

$$2\frac{\exp\left(-2\beta\right)}{r}(\alpha' + \beta') = 2\chi\frac{\partial\rho}{\partial\chi} \geq 0 \;. \tag{18}$$

In the following sections we will consider first a purely kinetic Lagrangian $\mathcal{L}(X)$ and then two Lagrangians $\mathcal{L} = f(\varphi)g(X)$ and $\mathcal{L} = g(X) - V(\varphi)$.

3. Unified Dark Matter models with purely kinetic Lagrangians

Let us consider a scalar field Lagrangian $\mathcal{L}$ with a non-canonical kinetic term that depends only on X or χ. Moreover, in this section we assume that $M_* = 0$ (or $M \gg M_*$).

First of all we must impose that $\mathcal{L}$ is negative when $X < 0$, so that the energy density is positive. Therefore, we define a new positive function

$$g_s(\chi) \equiv -\mathcal{L}(X) \ . \tag{19}$$

As already explained in Ref. [21], when the equation of state $p_\| = p_\|(\rho)$ is known, one can write the purely kinetic Lagrangian that describes this dark fluid with the help of Eqs. (6) and (8). Alternatively, using (13), one can connect $p_\|$ and ρ in terms of r through the variable R. Moreover, it is easy to see that starting from the field equation of motion (4), there exists another relation that connects χ (i.e. X) with r. This relation is

$$\chi \left[\frac{dg_s(\chi)}{d\chi}\right]^2 = \frac{k}{\left[r^2 \exp\alpha(r)\right]^2} \tag{20}$$

with k a positive constant. If we add an additive constant to $g_s(\chi)$, the solution (20) remains unchanged. One can see this also through the Eq. (13). Indeed, using Eqs. (6) and (8) one immediately finds that Eq. (13) is invariant under the transformation $\rho \to \rho + K \;\; p_\| \to p_\| - K$. In this way we can add the cosmological constant $K = \Lambda$ to the Lagrangian and we can describe the dark matter and the cosmological constant-like dark energy as a single dark fluid i.e. as Unified Dark Matter (UDM).

Let us notice that one can adopt two approaches to find reasonable rotation curves $v_c(r)$. A static solution can be studied in two possible ways:

i) The first approach consists simply in adopting directly a Langrangian that provides a viable cosmological UDM model and exploring what are the conditions under which it can give a static solution with a rotation curve that is flat at large radii. This prescription has been already applied, for example, in Ref. [21].

ii) A second approach consists in exploiting the invariance property of Eq. (15), with respect to the transformation (16) (when the condition (17) is satisfied). Usually in the literature one reduces the problem to the Newtonian gravity limit, because one makes use of a CDM density profile, i.e. one assumes that in Eq. (15), $p_\| \ll M/(4\pi r^3)$. We can therefore use Eqs. (16) and (17) to obtain energy density and pressure profiles $\rho(r)$ and $p_\|(r)$ that reproduce the same rotation curve in a model with non-negligible pressure. Next, we find an acceptable equation of state $p_\| = p_\|(\rho)$ such that we can reconstruct, through Eqs. (6) and (8), the expression for the Lagrangian $\mathcal{L}$. Such a procedure establishes a mapping between UDM and CDM solutions that predict the same halo rotation curve $v_c(r)$. As a starting point we could, of course, use very different CDM density profiles to this aim, such as the modified isothermal-law profile [31], the Burkert profile [32], the Moore profile [33], the Navarro-Frenk-White profile [34, 35] or the profile proposed by Salucci et al. (see for example [36]).

As we have already mentioned, the possible solutions one finds in this way have to satisfy the Einstein equations (10), (11) and (12), the conservation of stress-energy (13) and the weak energy conditions. Moreover, the resulting UDM scalar field Lagrangian must be able to provide cosmological solutions that yield an acceptable description of the cosmological background (see, e.g., Ref. [7]) and low effective speed of sound (see for example Refs. [2, 3, 6]) so that cosmic structure formation successfully takes place and CMB anisotropies fit the observed pattern [4, 5, 8].

Below, using approach i), we provide a worked example of a UDM model with purely kinetic Lagrangian which is able to describe a flat halo rotation curve and then, using approach ii), we give a general systematic procedure to obtain a possible Lagrangian of UDM model starting from a given CDM density profile.

3.1. Approach i): The generalized Scherrer solution

Let us consider the generalized Scherrer solution models obtained in Ref. [7]. These models are described by the following Lagrangian

$$\mathcal{L} = -\Lambda + g_n \left(X - \hat{X}\right)^n \tag{21}$$

where $g_n > 0$ is a suitable constant and $n > 1$. The case $n = 2$ corresponds to the unified model proposed by Scherrer [13]. If we impose that today $[(X - \hat{X})/\hat{X}]^n \ll 1$, the background energy density can be written as

$$\rho(a(t)) = \rho_\Lambda + \rho_{\rm DM} \ , \tag{22}$$

where ρ_Λ behaves like a "dark energy" component ($\rho_\Lambda = \text{const.}$) and $\rho_{\rm DM}$ behaves like a "dark matter" component i.e. $\rho_{\rm DM} \propto a^{-3}$, with $a(t)$ the scale factor.
A static solution for the generalized Scherrer model can be obtained in two possible ways:

1) Starting from the analysis of Ref. [15], in the case of a barotropic Lagrangian for the homogeneous field. The authors of Ref. [15] indeed concluded that for $n \gg 1$ flat halo rotation curves can be obtained. In particular they studied spherically symmetric solutions with the following metric,

$$ds^2 = -\left(\frac{r}{r_\star}\right)^l dt^2 + N(r)dr^2 + r^2 d\Omega^2. \tag{23}$$

where $r_\star$ is a suitable length-scale and $l = 2v_c^2$. In the trivial case where $N(r)$ is constant they find $\mathcal{L}(X) \propto X^{2/l}$ with $l \ll 1$. For $X \gg \hat{X}$ the Lagrangian $\mathcal{L} = -\Lambda + g_n(X - \hat{X})^n$ takes precisely this form.

2) In the analysis of Ref. [21], solutions where φ is only a function of the radius are considered. When the Lagrangian has the form $\mathcal{L} \propto X^n$, with $n \sim 10^6$ the halo rotation curve becomes flat at large radii. In this case n must be an odd natural number, such that the energy density is positive. Our model is able to reproduce this situation when the matter density is large, i.e. when $|X| \gg \hat{X}$.

Alternatively, if we wish to avoid large n (c.f. case 2) above) we can start from the following Lagrangian

$$\mathcal{L} = -\Lambda + \epsilon_X g_n \left(|X| - \hat{X}\right)^n \tag{24}$$

where ϵ_X is some differentiable function of X that is 1 when $X \geq \hat{X}$ and -1 when $X \leq -\hat{X} < 0$. In this way when $X > \hat{X} > 0$ we recover the Lagrangian of the generalized Scherrer solutions. When $X < 0$ and $\chi = -X > \hat{X}$ we get

$$\mathcal{L} = -\Lambda - g_n \left(\chi - \hat{X}\right)^n \tag{25}$$

and, with the help of Eqs. (6) and (8), we obtain

$$\rho = -p_\perp = -\mathcal{L} \,, \qquad p_\parallel = (2n-1) g_n \left(\chi - \hat{X}\right)^n + 2n g_n \hat{X} \left(\chi - \hat{X}\right)^{n-1} - \Lambda \,. \tag{26}$$

Now, requiring that χ be close to $\hat{X}$ (i.e. $\left(\chi - \hat{X}\right) \ll \hat{X}$) and $2n g_n \hat{X} \left(\chi - \hat{X}\right)^{n-1} \gg O(\Lambda)$, and starting from the relation (20) that connects χ with r, we get

$$\left(\chi - \hat{X}\right)^{n-1} = \frac{k^{1/2}}{n g_n \hat{X}^{1/2}} \frac{1}{r^2 \exp\left(\alpha(r)\right)} \,. \tag{27}$$

Consistency with our approximations implies that we have to consider the following expressions for radial configurations with r bigger than a minimum radius r_{min}. In this case $p_\parallel$ and ρ become

$$p_\parallel = \frac{A}{r^2 \exp\left(\alpha(r)\right)} \,, \qquad \rho = \frac{B}{\left[r^2 \exp\left(\alpha(r)\right)\right]^{n/(n-1)}} \tag{28}$$

where $A = 2(k\hat{X})^{1/2}$ and $B = g_n \left[k^{1/2}/(n g_n \hat{X}^{1/2})\right]^{n/(n-1)}$.
Using the Eqs. (10) and (11), we be able to calculate the values of the metric terms $\exp(\alpha)$ and $\exp(\beta)$ and, thus the value of ρ and $p_\parallel$. Alternatively we know that when $v_c \approx$ const. $\ll 1$ at large radii, in a first approximation, we can set $\exp(\alpha(r)) \approx C =$ const. Therefore for $n \neq 3$, we can write the function M as

$$M(r) \approx \frac{4\pi B}{C^{n/(n-1)}} \left(\frac{n-1}{n-3} r^{\frac{n-3}{n-1}} + D\right) \tag{29}$$

where we could also set $D = 0$ for $n > 3$. Instead, when $1 < n < 3$, the second term has to be larger than the first one.

In these cases v_c^2 becomes

$$v_c^2(r) \approx \frac{A}{2C} + \frac{B}{2\, C^{n/(n-1)}} \left(\frac{n-1}{n-3} \frac{1}{r^{2/(n-1)}} + \frac{D}{r}\right) \,. \tag{30}$$

For $n = 3$ we have

$$M(r) \approx \frac{4\pi B}{C^{3/2}} \ln\left(\frac{r}{\bar{r}}\right) + M(\bar{r}) \tag{31}$$

where $r > \bar{r}$ and

$$v_c^2 \approx \frac{A}{2C} + \frac{B}{2\, C^{3/2}} \frac{1}{r} \ln\left(\frac{r}{\bar{r}}\right) + \frac{M(\bar{r})}{8\pi r} \,. \tag{32}$$

In other words we see that the circular velocity becomes approximately constant for sufficiently large r.

However, let us stress that $\exp(\alpha(r))$ cannot be strictly constant, and that it should be chosen in such a way that the positivity of Eq. (12) is ensured.

This example can be generalized also to $M_* \neq 0$. Obviously, in such a case we have to assume that $r > r_b \geq r_{min}$. In this case k, r_{min}, A, B (through $\exp(\beta(r))$) and C depend on M_*.

The spherical top-hat solution for this model, which provides the link with the cosmological initial conditions, is described in the Appendix.

3.2. Approach ii): A general prescription to obtain UDM Lagrangians starting from a profile of an energy density distribution of CDM

Defining the energy density distribution of CDM as $\rho_{\rm CDM}(r)$ (with $p_{\rm CDM} = 0$), the transformation (16) becomes

$$\rho(r) = \rho_{\rm CDM}(r) + \sigma(r)\,, \qquad p_{\|}(r) = q(r)\,. \tag{33}$$

Now, starting from a given CDM density profile, through Eqs. (10), (11), (13) and (17) we can determine $\exp(\alpha)$, $\exp(\beta)$, ρ and $p_{\|}$. In a second step we provide the conditions to ensure that the energy density is positive ‡. In this case, after some simple but lenghty calculations, we find

$$\begin{aligned} &\mathcal{Q}'(r)\left(r\frac{M_{\rm CDM}(r)}{4\pi} - 2r\mathcal{Q}(r)\right) - 2\mathcal{Q}^2(r) \\ &+\mathcal{Q}(r)\left(4r + 3\frac{M_{\rm CDM}(r)}{4\pi} + 4r^3\rho_{\rm CDM}\right) = \frac{rM_{\rm CDM}(r)}{4\pi}\left(4 + 3r^2\rho_{\rm CDM}\right)\,, \end{aligned} \tag{34}$$

$$\mathcal{B}(r) = \mathcal{Q}(r) - \frac{M_{\rm CDM}(r)}{4\pi}\,, \tag{35}$$

$$\mathcal{A}(r) = \frac{\mathcal{Q}(r) + \mathcal{B}(r)}{2\mathcal{B}(r)}\,, \tag{36}$$

$$\sigma(r) = \frac{1 - \mathcal{Q}'(r)}{r^2} \tag{37}$$

where $\mathcal{Q}(r) = r(r^2 q + 1)$, $\mathcal{B}(r) = r\exp(-2\beta)$ and $\mathcal{A}(r) = (r\alpha' + 1)$. Here we define $M_{\rm CDM}(r) = 4\pi\int_0^r \tilde{r}^2 \rho_{\rm CDM}(\tilde{r})\, d\tilde{r}$. At this point it is easy to see that Eq. (34) does not admit a simple analytical solution for a generic $\rho_{\rm CDM}$. On the other hand we know that, through $\rho_{\rm CDM}$, all these functions depend on the velocity rotation curve $v_c(r)$. Moreover $v_c^2(r) \ll 1$. Therefore, defining $\bar{v}_c$ as the value that v_c assumes when the rotation curve is flat at large radii or the maximum value of v_c with a particular profile of $\rho_{\rm CDM}$, we can expand $\mathcal{Q}$, $\mathcal{A}$ and $\mathcal{B}$ as

$$\mathcal{Q}(r) = \mathcal{Q}_{(0)}(r) + \bar{v}_c^2\mathcal{Q}_{(1)}(r) + \frac{(\bar{v}_c^2)^2}{2!}\mathcal{Q}_{(2)}(r) + \ldots\,,$$

$$\mathcal{A}(r) = \mathcal{A}_{(0)}(r) + \bar{v}_c^2\mathcal{A}_{(1)}(r) + \frac{(\bar{v}_c^2)^2}{2!}\mathcal{A}_{(2)}(r) + \ldots\,,$$

‡ Thanks to this condition, through Einstein's Eq. (12), we can evade the *no-go* theorem derived in Ref. [15].

$$\mathcal{B}(r) = \mathcal{B}_{(0)}(r) + \bar{v}_c^2\mathcal{B}_{(1)}(r) + \frac{\left(\bar{v}_c^2\right)^2}{2!}\mathcal{B}_{(2)}(r) + \ldots \ . \tag{38}$$

Following this procedure we can determine ρ and $p_\|$ in a perturbative way, i.e.

$$\rho(r) = \rho_{(0)}(r) + \bar{v}_c^2\rho_{(1)}(r) + \frac{\left(\bar{v}_c^2\right)^2}{2!}\rho_{(2)}(r) + \ldots \ , \tag{39}$$

$$p_\|(r) = p_{\|\,(0)}(r) + \bar{v}_c^2 p_{\|\,(1)}(r) + \frac{\left(\bar{v}_c^2\right)^2}{2!}p_{\|\,(2)}(r) + \ldots \ . \tag{40}$$

Now, looking at the various CDM density profiles which have been proposed in the literature [31, 32, 33, 34, 35, 36], we see that we can always take $\rho_{\rm CDM}$ as

$$\rho_{\rm CDM}(r) = \bar{v}_c^2\rho_{\rm CDM\,(1)}(r) \ , \tag{41}$$

then

$$M_{\rm CDM}(r) = \bar{v}_c^2 M_{\rm CDM\,(1)}(r) = 4\pi\ \bar{v}_c^2\int_0^r \tilde{r}^2\rho_{\rm CDM\,(1)}(\tilde{r})\ d\tilde{r} \ . \tag{42}$$

For the zeroth-order terms we immediately obtain

$$\begin{aligned} \mathcal{Q}_{(0)} &= r \ , \\ \mathcal{A}_{(0)} &= 1 \ , \\ \mathcal{B}_{(0)} &= r \ . \end{aligned} \tag{43}$$

At the first order we get

$$\begin{aligned} \mathcal{Q}_{(1)} &= \frac{2}{r}\int_0^r \tilde{r}^3\rho_{\rm CDM\,(1)}(\tilde{r})\ d\tilde{r} \ , \\ \mathcal{A}_{(1)} &= \frac{1}{2r}\frac{M_{\rm CDM\,(1)}(r)}{4\pi} \ , \\ \mathcal{B}_{(1)} &= \frac{2}{r}\int_0^r \tilde{r}^3\rho_{\rm CDM\,(1)}(\tilde{r})\ d\tilde{r} - \frac{M_{\rm CDM\,(1)}(r)}{4\pi} \ . \end{aligned} \tag{44}$$

For completeness we write also the second order for $\mathcal{Q}$

$$\mathcal{Q}_{(2)} = \frac{1}{r}\int_0^r d\check{r}\frac{M_{\rm CDM\,(1)}(\check{r})}{4\pi}\left[\frac{2}{\check{r}}\mathcal{Q}_{(1)}(\check{r}) - \check{r}^2\rho_{\rm CDM\,(1)}(\check{r})\right] \ . \tag{45}$$

Let us stress that if one considers also terms $O\left(\bar{v}_c^4\right)$, Eq. (14) instead of Eq. (15) should be used. In such a case, v_c slightly changes with respect to the velocity rotation curve that one obtains using a CDM density profile.

For our purposes we can consider only the zeroth and the first-order terms. At this point, we can finally calculate the value of ρ and $p_\|$. We get

$$\rho(r) = \rho_{\rm CDM}(r) + \frac{1-\mathcal{Q}'(r)}{r^2} = \bar{v}_c^2\left(\frac{2}{r^4}\int_0^r \tilde{r}^3\rho_{\rm CDM\,(1)}(\tilde{r})\ d\tilde{r} - \rho_{\rm CDM\,(1)}(r)\right) \ , \tag{46}$$

$$p_\|(r) = \frac{\mathcal{Q}(r) - r}{r^3} = \bar{v}_c^2\frac{2}{r^4}\int_0^r \tilde{r}^3\rho_{\rm CDM\,(1)}(\tilde{r})\ d\tilde{r} \ . \tag{47}$$

As far as the values of the metric terms $\exp(\alpha)$ and $\exp(\beta)$ are concerned, we obtain the following expressions

$$\exp(2\alpha) = \exp(2\alpha(\hat{r}))\exp\left[\bar{v}_c^2\int_{\hat{r}}^r \frac{1}{\tilde{r}^2}\frac{M_{\rm CDM\,(1)}(\tilde{r})}{4\pi}\ d\tilde{r}\right] \tag{48}$$

$$\exp\left(-2\beta\right) = 1 + \frac{\bar{v}_c^2}{r^2}\left(2\int_0^r \tilde{r}^3 \rho_{\mathrm{CDM}\;(1)}(\tilde{r})\; d\tilde{r} - r\frac{M_{\mathrm{CDM}\;(1)}(r)}{4\pi}\right) . \tag{49}$$

Now, it is immediate to see that if we want a positive energy density we have to impose $2\ \int_0^r \tilde{r}^3 \rho_{\mathrm{CDM}\;(1)}(\tilde{r})\; d\tilde{r} \geq r^4\ \rho_{\mathrm{CDM}\;(1)}(r)$. From Eq. (10) we know that $M(r) = 4\pi\int_{\hat{r}_{(0)}}^r \tilde{r}^2 \rho(\tilde{r}) + M(\hat{r}_{(0)})$ and $M_{\mathrm{CDM}}(r) = 4\pi\int_{\bar{r}}^r \tilde{r}^2 \rho_{\mathrm{CDM}}(\tilde{r})\; d\tilde{r} + M_{\mathrm{CDM}}(\bar{r})$. Therefore we need to know what is the relation between $\bar{r}$ and $\hat{r}_{(0)}$. This condition is easily obtained if we make use of Eq. (15). Indeed, we get

$$\frac{M_{(1)}(\hat{r}_{(0)}) - M_{\mathrm{CDM}\;(1)}(\bar{r})}{4\pi} + \frac{2}{\hat{r}_{(0)}}\int_0^{\hat{r}_{(0)}} \tilde{r}^3 \rho_{\mathrm{CDM}\;(1)}(\tilde{r})\; d\tilde{r} = \int_{\bar{r}}^{\hat{r}_{(0)}} \tilde{r}^2 \rho_{\mathrm{CDM}\;(1)}(\tilde{r})\; d\tilde{r}\ , \tag{50}$$

which finally guarantees the invariance of the rotation velocity with respect to the transformation in Eqs. (16) and (17).

Let us, to a first approximation, parametrize the various CDM density profiles, at very large radii (i.e. when we can completely neglect the baryonic component) as

$$\rho_{\mathrm{CDM}} = \frac{\kappa\ \bar{v}_c^2}{r^n} \tag{51}$$

where κ is a proper positive constant which depends on the particular profile that is chosen [31, 32, 33, 34, 35, 36]. For example for many of the density profiles the slope is $n = 3$ for large radii [32, 33, 34, 35, 36].

In this case a positive energy density $\rho > 0$ requires $n \geq 2$. At this point let us focus on the case where $2 \leq n < 4$, since this gives rise to the typical slope of most of the density profiles studied in the literature. Therefore we obtain for $\rho(r)$ and $p_{\|}(r)$:

$$\rho(r) = \bar{v}_c^2\ \kappa\,\frac{n-2}{4-n}\frac{1}{r^n}\ , \qquad p_{\|}(r) = \bar{v}_c^2\ \kappa\,\frac{2}{4-n}\frac{1}{r^n}\ . \tag{52}$$

In particular,

1) for $n = 2$, we get

$$\rho(r) = 0, \qquad p_{\|}(r) = \rho_{\mathrm{CDM}} = \bar{v}_c^2\ \kappa\,\frac{1}{r^2}\ , \tag{53}$$

and for the relation between $\hat{r}_{(0)}$ and $\bar{r}$ one can choose, for example, $\hat{r}_{(0)} = \bar{r} = 0$. In other words, for large radii we have that $\rho(r) \ll p_{\|}(r)$.

2) Also for $2 < n < 3$ one can choose $\hat{r}_{(0)} = \bar{r} = 0$.

3) For $n = 3$

$$\rho(r) = \rho_{\mathrm{CDM}}, \qquad p_{\|}(r) = \bar{v}_c^2\ \kappa\,\frac{2}{r^3}\ , \tag{54}$$

and, through Eq. (50), we have to impose that

$$\frac{M_{(1)}(\hat{r}_{(0)}) - M_{\mathrm{CDM}\;(1)}(\bar{r})}{4\pi} = \ln\left(\frac{\hat{r}_{(0)}}{\bar{r}}\right) - 2\ . \tag{55}$$

Notice that the energy density profile is the same as the CDM one only for large radii so that $M_{(1)}(r)$ differs from $M_{\mathrm{CDM}\;(1)}(r)$.

4) In addition, for $3 < n < 4$, also through Eq. (50), we have to impose that

$$\frac{M_{(1)}(\hat{r}_{(0)}) - M_{\mathrm{CDM}\ (1)}(\bar{r})}{4\pi} = \frac{\bar{r}^{3-n}}{n-3} - \frac{(n-2)}{(4-n)(3-n)}\hat{r}_{(0)}^{3-n} . \tag{56}$$

Now let us focus where $2 < n < 4$. Starting from Eq. (52) to express $p_\parallel = p_\parallel(\rho)$ we solve Eq. (8) to recover the Lagrangian for the scalar field

$$\rho(\chi) = -\mathcal{L} = k\chi^{\frac{n}{2(n-2)}} , \qquad p(\chi) = \frac{2k}{(n-2)}\chi^{\frac{n}{2(n-2)}} \tag{57}$$

where k is a positive integration constant. We can see that, for this range of n, the exponent is larger than 1; thus there are no problems with a possible instability of the Lagrangian (see Refs. [21, 37, 38]). Therefore, through the transformation $\rho \to \rho + \Lambda$ $p_\parallel \to p_\parallel - \Lambda$, this Lagrangian can be extended to describe a unified model of dark matter and dark energy. Indeed, starting from the Lagrangian of the type (24), when $|X| \gg \hat{X}$ and if $k = g_n$, $\mathcal{L}$ takes precisely the form (57).

Finally, we want to stress that this prescription does not apply only to the case of an adiabatic fluid, such as the one provided by scalar field with a purely kinetic Lagrangian, but it can be also used for more general Lagrangians $\mathcal{L}(\varphi, X)$.

4. Unified Dark Matter models with non-purely kinetic Lagrangians

Let us consider more general Lagrangians of type $\mathcal{L} = \mathcal{L}(\varphi, X)$, with a non-canonical kinetic term, in order to find a UDM model with acceptable cosmological speed of sound. In this case we have one more degree of freedom: the scalar field configuration itself. Therefore, we have to impose a new condition to the solutions of the equation of motion. Ref. [7] required that the Lagrangian of the scalar field is constant along the classical trajectories. We want to know whether such a condition could be applied to the static spherically symmetric space-time metric. We would also like to know what the behavior of the rotation velocity v_c in the halo of a spiral galaxy is like for this class of models. In the next subsections we will consider first a Lagrangian of the form $\mathcal{L} = f(\varphi)g(X)$ and then a Lagrangian of the form $\mathcal{L} = g(X) - V(\varphi)$. In these cases, for simplicity, we will assume that $f(\varphi)$ and $V(\varphi)$ are positive.

4.1. Lagrangian of the type $\mathcal{L} = f(\varphi)g(X)$

Let us write the Lagrangian in the form $\mathcal{L} = f(\varphi)g(X) = -f(\varphi)g_s(\chi)$. Immediately we notice that the requirement of having a positive energy density imposes that $g_s(\chi)$ is positive. In this particular case the equation of motion (4) becomes

$$\frac{d}{dR}\left\{\ln\left|2\chi\frac{dg_s(\chi)}{d\chi} - g_s(\chi)\right|\right\} + 2\chi\frac{dg_s(\chi)}{d\chi}\left[2\chi\frac{dg_s(\chi)}{d\chi} - g_s(\chi)\right]^{-1} = -\frac{d\ln f(\varphi)}{dR} . \tag{58}$$

Moreover from Eq. (8) we obtain for $p_\parallel$

$$p_\parallel = f(\varphi)g_s(\chi)\left\{2\chi\frac{d\ln\left[g_s(\chi)\right]}{d\chi} - 1\right\} . \tag{59}$$

Following the procedure previously explained we impose the constraint $\mathcal{L} = -\rho = -\Lambda$, i.e.

$$f(\varphi) = \frac{\Lambda}{g_s(\chi)} \, , \tag{60}$$

which, inserted in the equation of motion (58), allows to find the following general solution

$$\chi \frac{d \ln [g_s(\chi)]}{d\chi} = \frac{k/2}{r^2 \exp(\alpha)} \, , \tag{61}$$

where k is a constant of integration. Now, inserting Eqs. (60) and (61) into the relation (59), we obtain

$$p_{\|} = \frac{\Lambda k}{r^2 \exp(\alpha)} - \Lambda \, . \tag{62}$$

Using this expression and considering the halo for $r > r_b$ and $M \gg M_*$, we are finally able to get the expression for the rotation velocity

$$v_c^2 \simeq \frac{\Lambda k/2}{\exp(\alpha)} - \frac{\Lambda r^2}{3} \, . \tag{63}$$

If $\exp(\alpha) \approx$ const. this expression leads to a flat rotation curve for all radii $r < r_{\max}$ such that $r_{\max}^2 \ll 3k/(2\exp(\alpha))$ and provided that the constant k is positive. Therefore, in the future we will always neglect the second term in Eq. (63).

It is important to stress that the results outlined in Eqs. (60)-(63) give an efficient recipe to obtain a flat halo rotation curve within the UDM scenario. Once a Langrangian (i.e. $g(X)$) leading to a viable UDM model on cosmological scales is obtained by imposing the constraint $\mathcal{L} = -\Lambda$ (see Ref. [7]), a flat rotation curve is guaranteed through Eq.(63). There are however two important requirements that have to be satisfied. The function $g_s(\chi)$ must allow for a positive integration constant k through Eq.(61), and the Lagrangian must satisfy the stability conditions discussed in Refs. [21, 37, 38]), which require $\partial\mathcal{L}/\partial X > 0$ and $\partial\mathcal{L}/\partial X + 2X\partial^2\mathcal{L}/\partial X^2 > 0$ (so that the speed of sound is positive both in the cosmological setting and for the static solution).

In the second part of this subsection we will consider first a situation where $M_* = 0$, in other words when our halo is composed only of the dark fluid, and then a situation where there is a non-negligible baryon contribution in the inner part of the halo.

4.1.1. Case $M_* = 0$: halo composed only of the dark fluid Starting from $\rho = \Lambda$ and $p_{\|} = \Lambda k/(r^2 \exp(\alpha)) - \Lambda$ we can explicitly calculate the value of $\exp(\alpha)$ and $\exp(\beta)$ through Eqs. (10) and (11). Therefore, for $M_* = 0$ we get

$$\exp(-2\beta) = 1 - \frac{\Lambda r^2}{3} \, , \tag{64}$$

$$\exp(\alpha) = \frac{\Lambda k}{2} \left\{ \left(1 - \frac{\Lambda r^2}{3}\right)^{1/2} \left[\frac{2\kappa}{\Lambda k} - \ln \frac{\left(\frac{\Lambda}{3}\right)^{1/2} r}{1 - \left(1 - \frac{\Lambda r^2}{3}\right)^{1/2}} \right] + 1 \right\} \, , \tag{65}$$

where κ is a suitable positive integration constant. In particular, the value of κ should be such that the term on the RHS of Eq. (65) is positive, i.e.

$$\left(\frac{\Lambda}{3}\right)^{1/2} r > \left[\cosh\left(\frac{2\kappa}{\Lambda k}+1\right)\right]^{-1} . \tag{66}$$

It is very important to stress that, in this case, the weak energy conditions are satisfied. In other words, through this prescription, we are able to evade the *no-go* theorem derived in Ref. [15].

Using Eq. (65) and Eq. (63) we can obtain the following expression for the circular velocity

$$v_c^2(r) = \left\{\left(1-\frac{\Lambda r^2}{3}\right)^{1/2}\left[\frac{2\kappa}{\Lambda k} - \ln\frac{\left(\frac{\Lambda}{3}\right)^{1/2} r}{1-\left(1-\frac{\Lambda r^2}{3}\right)^{1/2}}\right]+1\right\}^{-1} . \tag{67}$$

In order to have values of $v_c \sim 10^{-3}$ we must impose that $2\kappa/(\Lambda k) \sim 10^6 \ll 3/(\Lambda r_{max}^2)$. Imposing this condition we can obtain an approximately flat halo rotation curve.

A simple inspection of Eqs. (65), (66) and (67) shows an interesting property of our result. There is a minimum radius $r_{min} \approx (\Lambda/3)^{-1/2}\left[\cosh\left(2\kappa/(\Lambda k)\right)\right]^{-1}$ required for the validity of (66). Obviously it is necessary that $r_{min} \ll r_{gal}$ ($\ll r_{max}$) where r_{gal} is the typical radius of our halo.

4.1.2. Case $M_* \neq 0$: non-negligible baryonic component in the center of the halo

In this subsection we assume that $r > r_b$. If $M \gg M_*$ we recover the same result of the previous subsection; if $M_* \gg O(\Lambda r^3)$, using Eqs. (10) and (11), we obtain

$$\exp(-2\beta) \approx 1 - \frac{M_*}{4\pi r} \tag{68}$$

$$\exp(\alpha) \approx \Lambda k\left\{\left(1-\frac{M_*}{4\pi r}\right)^{1/2}\left[\frac{\kappa_*}{\Lambda k} + \cosh^{-1}\left(\frac{4\pi r}{M_*}\right)^{1/2}\right] - 1\right\} . \tag{69}$$

where κ_* is a suitable positive integration constant. In particular it easy to see that κ_* and k (through $\exp(\beta(r))$) depend also on the value of M_*, since one is considering $r > r_b$. Obviously, these functions exist only for $r > r_* > M_*/(4\pi)$, having defined r_* as the value of the radius for which $\exp(\alpha(r_*)) = 0$. In this case, using the approximate relation (15), v_c reads

$$v_c^2 \approx \frac{1}{2}\left\{\left(1-\frac{M_*}{4\pi r}\right)^{1/2}\left[\frac{\kappa_*}{\Lambda k} + \cosh^{-1}\left(\frac{4\pi r}{M_*}\right)^{1/2}\right] - 1\right\}^{-1} + \frac{M_*}{8\pi r} . \tag{70}$$

To have halo rotation velocities $v_c \sim 10^{-3}$ for $r \gg r_*$, we need to impose $2\kappa_*/(\Lambda k) \sim 10^6$. One can see that this condition leads to $r_* \approx M_*/(4\pi)$. Moreover, also in this case we have a minimum radius r_{min} such that $v_c^2(r_{min}) = 1$. Starting from Eq. (14) we get

$$r_{min} \approx \frac{M_*}{2\pi} > r_* . \tag{71}$$

4.2. Lagrangian of the type $\mathcal{L} = g(X) - V(\varphi)$

In this subsection we briefly discuss Lagrangians of the type $\mathcal{L} = g(X) - V(\varphi)$. Let us rewrite $\mathcal{L}$ as $\mathcal{L} = -[g_s(\chi) + V(\varphi)]$. In order to have $\rho > 0$ we impose that $g_s(\chi) > 0$. For these Lagrangians the equation of motion (4) becomes

$$\chi \frac{dg_s(\chi)}{d\chi} \left\{ \frac{d \ln \left[\chi \left(\frac{dg_s(\chi)}{d\chi} \right)^2 \right]}{dR} + 2 \right\} = \frac{d \ln V(\varphi)}{dR} \, . \tag{72}$$

Requiring that the Lagrangian of the scalar field is constant along the classical trajectory, i.e

$$V(\varphi) = -g_s(\chi) + \Lambda \, , \tag{73}$$

from the Eq. (72) we get

$$\chi \frac{dg_s(\chi)}{d\chi} = \frac{k/2}{r^2 \exp \alpha} \tag{74}$$

where k is a positive constant.
Now, inserting Eq. (74) into Eq. (8) we obtain the same expressions for $p_{\|}$ and v_c that we obtained in the last subsection i.e. Eqs. (62) and (63), respectively.

5. Conclusions

In this paper we have investigated static spherically symmetric solutions ("dark halos") of Einstein's equations for a scalar field with non-canonical kinetic term. Assuming that the scalar field depends only on the radius, we studied Unified Dark Matter models with purely kinetic Lagrangians. In particular, we obtained a purely kinetic Lagrangian which allows simultaneously to produce flat halo rotation curves and to realize a unified model of dark matter and dark energy on cosmological scales. Moreover, we gave a prescription to obtain UDM model solutions that have the same rotation curve $v_c(r)$ as a CDM model with a specified density profile. Next, we considered a more general class of Lagrangians with non-canonical kinetic term. In this case we have one more degree of freedom (the scalar field configuration itself) and we need to impose one more constraint. To this aim, we required that the Lagrangian is constant, $\mathcal{L} = -\Lambda$ along the solutions of the equation of motion. We have studied whether this condition can be applied to the static spherically symmetric space-time metric and what the behavior of $v_c(r)$ is for this class of models.

Let us finally stress that these solutions allow for the possibility to find suitable Lagrangians that describe with a single fluid viable cosmological and static solutions.

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Appendix A. Spherical collapse for purely kinetic scalar-field Lagrangians

Let us assume a flat, homogeneous Friedmann-Robertson-Walker background metric i.e.

$$ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j \,, \tag{A.1}$$

where $a(t)$ is the scale factor, δ_{ij} denotes the unit tensor. Moreover, for this particular case, the Hubble parameter H is a function only of the UDM fluid $H^2 = \rho/3$.
Now let us consider a top-hat spherical overdensity with the purely kinetic model with the Lagrangian $\mathcal{L} = -\Lambda + g_n(X - \hat{X})^n$ and with $g_n > 0$ [7]. For this particular case within the overdense region we have a single dark fluid ondergoing spherical collapse, which is described by the following equation

$$\frac{\ddot{R}}{R} = -\frac{1}{6}\left(\rho_R + 3p_R\right) \tag{A.2}$$

where $R(t)$, ρ_R and p_R are respectively the scale-factor, pressure and energy density of the overdense region and where the dot denotes differentiation w.r.t. the cosmic time t.
Now, ρ_R and p_R are defined by the following expressions [7]

$$\rho_R = \Lambda + 2ng_n\hat{X}(X_R - \hat{X})^{n-1} + (2n-1)g_n(X_R - \hat{X})^n \tag{A.3}$$

$$p_R = g_R = -\Lambda + g_n(X_R - \hat{X})^n \tag{A.4}$$

with $X_R = X(R)$ a function of time.

The equation of motion is

$$\left(\frac{\partial g_R}{\partial X_R} + 2X\frac{\partial^2 g_R}{\partial X_R^2}\right)\frac{dX_R}{dN_R} + 3\left(2X_R\frac{\partial g_R}{\partial X_R}\right) = 0 \,. \tag{A.5}$$

where $dN_R = dR/R$. The solution of Eq. (A.5) (for $\partial g_R/\partial X_R, X_R \neq 0$) is

$$X_R\left(\frac{\partial g_R}{\partial X_R}\right)^2 = k_R R^{-6} \tag{A.6}$$

where we can choose $k_R = R_{ta}^6\left[X_R\left(\frac{\partial g_R}{\partial X_R}\right)^2\right]_{ta}$, with R_{ta} the value of R at turnaround.
Replacing Eq. (A.4) in Eq. (A.6) we find

$$X_R\,\left[ng_n(X_R - \hat{X})^{n-1}\right]^2 = k_R R^{-6} \tag{A.7}$$

Using now the explicit expressions for ρ_R and p_R we arrive at the following set of equations

$$\frac{\ddot{R}}{R} = -\frac{1}{3}\left[-\Lambda + ng_n\hat{X}(X_R - \hat{X})^{n-1} + (n+1)g_n(X_R - \hat{X})^n\right] \tag{A.8}$$

$$(X_R - \hat{X})^{2n-1} + \hat{X}(X_R - \hat{X})^{2(n-1)} = \frac{k_R}{n^2 g_n^2}R^{-6}. \tag{A.9}$$

For $(X_R - \hat{X})/\hat{X} \ll 1$ Eq. (A.8) becomes

$$\frac{\ddot{R}}{R} = -\frac{1}{3}\left\{-\Lambda + ng_n|X_{R_{ta}} - \hat{X}|^{n-1}(X_{R_{ta}}\hat{X})^{\frac{1}{2}}\left(\frac{R}{R_{ta}}\right)^{-3}\right\} \tag{A.10}$$

We can now write all the equations that describe the spherical collapse

$$
\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{1}{3}\left(\rho_\Lambda + \rho_{\rm DM}\right) && \text{(A.11)}\\
\rho_\Lambda &= \Lambda && \text{(A.12)}\\
\rho_{\rm DM} &= 2ng_n|X_{ta}-\hat{X}|^{n-1}(X_{ta}\hat{X})^{\frac{1}{2}}\left(\frac{a}{a_{ta}}\right)^{-3} && \text{(A.13)}\\
\frac{\ddot{R}}{R} &= -\frac{1}{6}\left(\rho_{R_{DM}} - 2\rho_{R_\Lambda}\right) && \text{(A.14)}\\
\rho_{R_{\rm DM}} &= 2ng_n|X_{ta}-\hat{X}|^{n-1}(X_{R_{ta}}\hat{X})^{\frac{1}{2}}\left(\frac{R}{R_{ta}}\right)^{-3} && \text{(A.15)}
\end{aligned}
$$

where $a_{ta} = a(t_{ta})$.

Following now the same procedure of Ref. [39] we can define x and y

$$
\begin{aligned}
x &\equiv \frac{a}{a_{ta}} && \text{(A.16)}\\
y &\equiv \frac{R}{R_{ta}}\ . && \text{(A.17)}
\end{aligned}
$$

In this way we can redefine $\rho_{\rm DM}$ and $\rho_{R_{\rm DM}}$ such that

$$
\begin{aligned}
\rho_{\rm DM} &= \frac{3H_{ta}^2\Omega_{\rm DM}(x=1)}{x^3} && \text{(A.18)}\\
\rho_{R_{\rm DM}} &= \zeta\frac{3H_{ta}^2\Omega_{\rm DM}(x=1)}{y^3} && \text{(A.19)}
\end{aligned}
$$

where $\Omega_{\rm DM}$ is the (k-essence) dark matter density parameter, and $\zeta = (\rho/\rho_{\rm DM})|_{x=1}$. Then Eqs. (A.11) and (A.14) become

$$
\begin{aligned}
\frac{dx}{d\tau} &= \left(x\Omega_{DM}(x)\right)^{-\frac{1}{2}}\ , && \text{(A.20)}\\
\frac{d^2y}{d\tau^2} &= -\frac{1}{2y^2}\left[\zeta - 2y^3K_\Lambda\right]\ , && \text{(A.21)}\\
\Omega_{DM}(x) &= \left(1-\frac{1-\Omega_{DM}(x=1)}{\Omega_{DM}(x=1)}x^3\right)^{-1}\ , && \text{(A.22)}
\end{aligned}
$$

where $d\tau = H_{ta}\sqrt{\Omega_{DM}(x=1)}$ and $K_\Lambda = \rho_\Lambda/[3H_{ta}^2\Omega_{DM}(x=1)]$.

Defining U as the potential energy of the overdensity and using energy conservation between virialization and turnaround,

$$
\left[U + \frac{R}{2}\frac{\partial U}{\partial R}\right]_{vir} = U_{ta}\ , \tag{A.23}
$$

we obtain

$$
(1+q)y - 2qy^3 = \frac{1}{2} \tag{A.24}
$$

where

$$
q = \left(\frac{\rho_\Lambda}{\rho}\right)_{y=1} = \frac{K_\Lambda}{\zeta}\ , \tag{A.25}
$$

in full agreement with Ref. [40].

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